

Wednesday 4th May 2005, 10 am

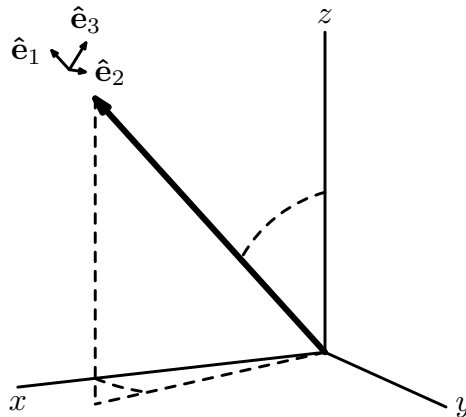
Time Allowed: 3 hours

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given and the beginning of each Section.

Calculators are not permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Section A: *You should attempt all questions. Marks awarded are shown next to the questions.*

1. Consider the sphere $x^2 + y^2 + z^2 + 6x + 8y = 0$.
 - (a) Find a plane tangent to the sphere at the origin.
 - (b) Write down the position vector for a line through the origin and normal to the sphere.
 - (c) Verify that the line passes through the centre of the sphere. [10]
2. The arrows labelled $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ below indicate basis vectors in a standard coordinate system.



Identify the basis vectors, including minus signs if needed. Justify your identification briefly. [6]

3. Using integral theorems, simplify the following:
 - (a) $\oint \mathbf{r} \cdot d\mathbf{S}$ over an arbitrary closed surface,
 - (b) $\oint \mathbf{r} \cdot d\mathbf{r}$ over an arbitrary closed path,
 - (c) $\oint d\mathbf{S}$ over an arbitrary closed surface. [12]

4. Using index notation, or otherwise, show that

$$|\mathbf{A} \times \mathbf{B}|^2 = |\mathbf{A}|^2 |\mathbf{B}|^2 - |\mathbf{A} \cdot \mathbf{B}|^2.$$

You may assume standard index identities. [12]

5. Express the following as Fourier series.

(a) $\sin(x + \frac{\pi}{4})$

(b) $\sin^4 x$

It is not necessary to do any integrals. [6]

6. Given that $\Phi(\rho, \phi) = \rho^k \cos m\phi$ (where k and m are constants) satisfies Laplace's equation

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

find k in terms of m . [6]

7. We want to find a path $y(x)$ between two given endpoints such that

$$\int (y' + y)^2 dx$$

is extremized. [Here y' means dy/dx as usual.] Determine the Euler-Lagrange equation for the problem and its general solution.

What is the particular solution if the endpoints are $(1, 0)$ and $(0, 1)$? [8]

Section B: Each question carries 20 marks. You may attempt all questions but only marks for the best TWO questions will be counted.

8. Show that

$$f(x) = \begin{cases} x(\pi - x) & \text{for } x > 0 \\ x(\pi + x) & \text{for } x < 0 \end{cases}$$

has Fourier expansion

$$\sum_{n \text{ odd}}^{\infty} \frac{8}{n^3 \pi} \sin(nx)$$

for $x \in [-\pi, \pi]$.

You may assume standard results on Fourier series, with brief justification.

9. Consider the two-dimensional coordinate system (α, β) defined through

$$x = \sinh \alpha \sin \beta, \quad y = \cosh \alpha \cos \beta.$$

Show that this coordinate system is orthogonal.

10. A cylindrical apple corer of radius c cuts through a spherical apple of radius a . Write down the cylindrical coordinates ρ and z at which the corer meets the peel. Hence or otherwise show that the volume of apple remaining is

$$\frac{4\pi}{3} (a^2 - c^2)^{3/2}.$$

You may use standard results on coordinate systems.

11. Given two well-behaved functions $u(x, y)$ and $v(x, y)$ satisfying

$$\left(\frac{\partial u}{\partial x} \right) = \left(\frac{\partial v}{\partial y} \right), \quad \left(\frac{\partial u}{\partial y} \right) = - \left(\frac{\partial v}{\partial x} \right)$$

and an arbitrary closed curve C in the x, y plane, show that

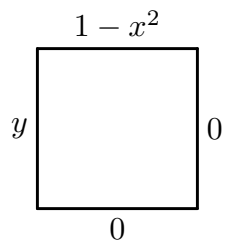
$$\oint_C (u + iv) (dx + idy) = 0$$

is a consequence of Stokes' theorem.

12. The function $F(x, y)$ satisfies Laplace's equation in the square $x, y \in [0, 1]$ and has boundary conditions

$$\begin{cases} F(x, 1) = 1 - x^2 \\ F(0, y) = y \\ F(x, 0) = F(1, y) = 0 \end{cases}$$

that is to say



Show that

$$F(x, y) = y - xy + \sum_{n \text{ odd}} \frac{8}{n^3 \pi^3 \sinh(n\pi)} \sin(n\pi x) \sinh(n\pi y).$$

[You may assume the formula given in Question 8.]

End of Examination

P. Saha

Calculus III

Answers

1. {Similar problems seen.}

(a) Taking the gradient gives the normal to the surface, at the origin

$$\mathbf{p} = 3\mathbf{i} + 4\mathbf{j}. \quad [2]$$

The tangent plane at the origin is $\mathbf{r} \cdot \mathbf{p} = 0$ which simplifies to

$$3x + 4y = 0. \quad [3]$$

(b) $\mathbf{r} = s(3\mathbf{i} + 4\mathbf{j})$ where $-\infty < s < \infty$. [2]

(c) The given sphere can be rewritten as

$$(x - 3)^2 + (y - 4)^2 + z^2 = 25 \quad [1]$$

and $\mathbf{r} = -(3\mathbf{i} + 4\mathbf{j})$ is the centre. [2]

2. {Variant of coursework problem}

We recognize spherical polar basis vectors. Specifically:

($\hat{\mathbf{e}}_1$) points towards increasing r and hence is $\hat{\mathbf{e}}_r$, [2]

($\hat{\mathbf{e}}_2$) points towards increasing ϕ and hence is $\hat{\mathbf{e}}_\phi$, [2]

($\hat{\mathbf{e}}_3$) points towards decreasing θ and hence is $-\hat{\mathbf{e}}_\theta$. [2]

3. {Coursework problems}

(a) Using the divergence theorem

$$\oint \mathbf{r} \cdot d\mathbf{S} = \int (\nabla \cdot \mathbf{r}) dV \quad [2]$$

resulting in 3 times the volume enclosed by the surface. [2]

(b) Using Stokes' theorem

$$\oint \mathbf{r} \cdot d\mathbf{r} = \int (\nabla \times \mathbf{r}) \cdot d\mathbf{S} \quad [2]$$

giving zero. [2]

(c) Consider the $\mathbf{k}$ component:

$$\mathbf{k} \cdot \oint d\mathbf{S} = \int (\nabla \cdot \mathbf{k}) dV \quad [3]$$

giving zero. [1]

4. {Special case of coursework problem.}

In index notation, $|\mathbf{A} \times \mathbf{B}|$ is

$$\epsilon_{ijk} A_j B_k \quad [2]$$

and the LHS becomes

$$\epsilon_{ijk} A_j B_k \epsilon_{imn} A_m B_n \quad [2]$$

which the standard two- ϵ identity changes to

$$(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})A_j B_k A_m B_n. \quad [3]$$

Simplifying, we get

$$A_j B_k A_j B_k - A_j B_k A_k B_j \quad [3]$$

which we identify as the required RHS. [2]

5. {Other examples of trivial Fourier series seen.}

We need to realize that any valid sin/cos series answers the stated question. [1]

$$\begin{aligned} \text{(a)} \quad \sin(x + \frac{\pi}{4}) &= \frac{1}{\sqrt{2}} \cos x + \frac{1}{\sqrt{2}} \sin x \\ &\text{or } \sin u \text{ where } u = x + \frac{\pi}{4} \end{aligned} \quad [2]$$

$$\text{(b)} \quad \sin^4 x = \frac{1}{4}(1 - \cos 2x)^2 = \frac{5}{8} - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x \quad [3]$$

6. {Other variable-separation examples covered.}

We substitute into the given equation

$$\frac{\cos m\phi}{\rho} \frac{d}{d\rho} \left(\rho \frac{d\rho^k}{d\rho} \right) - \rho^{k-2} m^2 \cos m\phi = 0 \quad [3]$$

and simplify to get $k^2 = m^2$. [3]

7. {Similar problems in notes and coursework.}

We work out the Euler-Lagrange equation

$$\frac{dy'}{dx} - y = 0$$

or

$$y'' - y = 0 \quad [3]$$

The well-known general solution is

$$y = A \cosh x + B \sinh x \quad [3]$$

Boundary conditions: the second point fixes $A = 1$ and the other boundary condition gives $B = -\coth 1$. Hence

$$y = \cosh x - \frac{\sinh x}{\tanh 1} \quad [2]$$

8. {Seen, but slightly differently stated, in notes and coursework.}

We see that $f(-x) = -f(x)$ (odd function). [2]

So we use the standard formula for odd functions

$$b_n = \frac{2}{\pi} \int_0^\pi x(\pi - x) \sin nx \, dx. \quad [3]$$

{Formula flagged in notes as to be learned.}

Integrating by parts

$$\begin{aligned} & \int_0^\pi x(\pi - x) \sin nx \, dx \\ &= \left| -x(\pi - x) \frac{\cos nx}{n} \right|_0^\pi \{ \text{gives zero} \} + \int_0^\pi (\pi - 2x) \frac{\cos nx}{n} \, dx \end{aligned} \quad [4]$$

$$= \left| (\pi - 2x) \frac{\sin nx}{n^2} \right|_0^\pi \{ \text{gives zero} \} + \int_0^\pi 2 \frac{\sin nx}{n^2} \, dx \quad [4]$$

$$= -\frac{2}{n^3} |\cos nx|_0^\pi \quad [3]$$

$$= \frac{4}{n^3} \quad (n \text{ odd}) \quad [4]$$

which is the expected result.

{A fairly mechanical question, but it does test retention of elementary calculus.}

9. {Seen in coursework}

Orthogonality means $\hat{\mathbf{e}}_\alpha \cdot \hat{\mathbf{e}}_\beta = 0$. [2]

Since by definition

$$\hat{\mathbf{e}}_\alpha = \frac{\frac{\partial \mathbf{r}}{\partial \alpha}}{\left| \frac{\partial \mathbf{r}}{\partial \alpha} \right|} \quad \hat{\mathbf{e}}_\beta = \frac{\frac{\partial \mathbf{r}}{\partial \beta}}{\left| \frac{\partial \mathbf{r}}{\partial \beta} \right|} \quad [3]$$

it is enough to show that

$$\left(\frac{\partial \mathbf{r}}{\partial \alpha} \right) \cdot \left(\frac{\partial \mathbf{r}}{\partial \beta} \right) = 0. \quad [2]$$

We start from

$$\mathbf{r} = \sinh \alpha \sin \beta \mathbf{i} + \cosh \alpha \cos \beta \mathbf{j} \quad [4]$$

and get

$$\left(\frac{\partial \mathbf{r}}{\partial \alpha} \right) = \cosh \alpha \sin \beta \mathbf{i} + \sinh \alpha \cos \beta \mathbf{j} \quad [3]$$

$$\left(\frac{\partial \mathbf{r}}{\partial \beta} \right) = \sinh \alpha \cos \beta \mathbf{i} - \cosh \alpha \sin \beta \mathbf{j} \quad [3]$$

and working out the dot product gives zero. [3]

{Easy for someone who has understood what basis vectors and orthogonality mean.}

10. {Seen in notes for $a = 1$.}

The corer always has $\rho = c$ and it meets the peel at $z = \pm\sqrt{a^2 - c^2}$. [4]

The volume element in cylindrical coordinates is $\rho d\rho d\phi dz$ and the integral over ϕ trivially gives 2π . [4]

That leaves us to calculate

$$2\pi \int_{-\sqrt{a^2-c^2}}^{\sqrt{a^2-c^2}} dz \int_c^{\sqrt{a^2-z^2}} \rho d\rho \quad [5]$$

$$= \pi \int_{-\sqrt{a^2-c^2}}^{\sqrt{a^2-c^2}} (a^2 - c^2 - z^2) dz = \pi \left| (a^2 - c^2)z - \frac{1}{3}z^3 \right|_{-\sqrt{a^2-c^2}}^{\sqrt{a^2-c^2}} \quad [4]$$

$$= \frac{4\pi}{3} (a^2 - c^2)^{3/2}. \quad [3]$$

11. {Stokes' theorem $\Rightarrow$ Green's theorem is bookwork, rest was a coursework problem.}

First we consider

$$\oint_C (X \mathbf{i} + Y \mathbf{j}) \cdot d\mathbf{r} = \oint_C (X dx + Y dy). \quad [4]$$

Provided X, Y are well-behaved, we can apply Stokes' theorem to get

$$\int \nabla \times (X \mathbf{i} + Y \mathbf{j}) dS = \int \int \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) dx dy \quad [4]$$

where the integral is over the area enclosed by C . {Green's theorem}

The real and imaginary parts of the integral are

$$\int (u dx - v dy) + i \int (v dx + u dy). \quad [4]$$

Apply the above identity gives

$$- \int \int \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dx dy + i \int \int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy. \quad [4]$$

The given relation between the derivatives makes both integrands zero. [4]

12. {Variant of coursework problems}

We express $F(x, y)$ as

$$F(x, y) = A + Bx + Cy + Dxy + G(x, y)$$

where A, B, C, D are constant and G is zero at the corners. [3]

Solving at the corners we get

$$F(x, y) = y - xy + G(x, y) \quad [3]$$

where $G(x, y)$ will have boundary conditions

$$\begin{array}{c} x(1-x) \\ \square \\ 0 \qquad \qquad 0 \\ 0 \end{array} \quad [3]$$

A solution of the form

$$G(x, y) = \sum_n C_n \sin(n\pi x) \sinh(n\pi y)$$

will satisfy Laplace's equation and be zero on three sides. [3]

To satisfy the remaining boundary condition we need

$$\sum_n C_n \sinh(n\pi) \sin(n\pi x) = x(1-x). \quad [3]$$

From Question 8 we have

$$x(\pi - x) = \sum_{n \text{ odd}}^{\infty} \frac{8}{n^3 \pi} \sin(nx)$$

and replacing x by πx we have

$$x(1-x) = \sum_{n \text{ odd}}^{\infty} \frac{8}{n^3 \pi^3} \sin(n\pi x). \quad [3]$$

Comparing coefficients we get

$$C_n = \frac{8}{n^3 \pi^3 \sinh(n\pi)} \quad (n \text{ odd}). \quad [2]$$